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Hence a specialist in geometry may confine himself within just as narrow limits as a specialist in arithmetic and algebra may do. Since analysis is only algebra grown big, it is clear that it is more directly dependent upon the preceding grand division than geometry is.

As mathemathetics largely consists in going from one concept to a related concept, and as concepts which involve the most extensive intimate relations with others are the most helpful, it is clear that it is just as important at times to be able to overlook details as it is at other times to obtain interesting illuminating results from a study of details. For instance, Hamilton, in his lectures on quaternions, makes a distinction between operator and operandus which is unnecessary for applications and is also a hindrance to clear exposition.* One of the great advantages of abstract group theory rests upon overlooking the distinction between operator and operand in many applications. The distinction between a system of conjugate substitutions and a group of permutations as used by Cauchy and Serret is also of doubtful value. The mathematician must be able to leap from mountain top to mountain top as well as to dig out the gold from old river beds concealed by the work of geologic ages.

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A CURIOUS MECHANICAL PARADOX.

By EDWIN BIDWELL WILSON, Massachusetts Institute of Technology.

Whether a paradox appears merely as a pest depends largely on the point of view. If the paradox lies in some well known and thoroughly accredited discipline such as elementary algebra, geometry, or mechanics, it is certainly a nuisance except in so far as it may be pedagogically instructive for the purpose of reinforcing, by its solution, the very principles which it would upset. To this class belong the demonstrations that $2=4$, usually dependent on a division by 0 or a carelessly placed sign in the extraction of a square root, and the proof that all triangles are isosceles or all angles right angles, dependent on incorrectly drawn figures, and finally the proposition that a rolling wheel must be lighter than one at rest owing to the resultant centrifugal force attributable to the rotation of the wheel about its instantaneous center. Such paradoxes are even a hindrance pedagogically rather than a help unless the fallacy involved can be made much clearer than the fallacious demonstration — a thing almost impossible to accomplish unless the student is so well gounded in the fundamentals of the science that he would not himself fall into like errors. On the other hand, when the paradox arises in a field which is not yet familiar even to specialists in it, a thorough

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*E. Study, *Encyklopaedie der Mathematischen Wissenschaften*, vol. I, page 159.

examination of the paradox may be of first rate scientific importance. The attention which eminent men have paid to the contradictions developable in the theory of transfinite numbers and in the early discussions of the second law of thermodynamics might be mentioned as illustrations. It is doubtful if all the paradoxes connected with kinetic theory are yet cleared away from the statistical study of the interactions of matter and ether; and from the point of view of the new theory of relativity the Lorentz shortening of a body in its line of motion may easily take on the aspect of a paradox.

A curious and withal instructive example of the former kind of paradox has just been brought to my attention and seems worthy of solution not only for those who have been puzzled by it but for those who may never have thought of it. Here it is — problem, paradox, and solution.

Problem. Suppose a tank of water with an efflux pipe, which terminates vertically, is resting on a frictionless plane when the water begins to flow. What is the motion of the system?

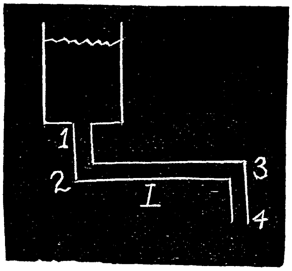

Fig. 1.

To make the problem as specific as possible and at the same time to remove any unnecessary lack of symmetry, let it be assumed that the efflux pipe 1234 starts from the center of the bottom of the tank, runs through a horizontal distance L and ends with a vertical section as in the figure. It is this vertical ending which introduces the paradox.

Paradox. Let the water begin to flow out. As there are no horizontal external forces acting on the system, the center of gravity of the whole system must remain in the same vertical line. Hence the tank must slide to the left as the water is carried to the right. But as the tank slides to the left the water which leaves at 4 has a horizontal component toward the left and again, as there is no external horizontal force, the water will keep on indefinitely moving toward the left which is the same direction that the tank moves. Hence the center of gravity of the system will ultimately be found far to the left of its original position instead of in the same vertical line.

Solution. The tank does start moving toward the left and a certain amount of water will move indefinitely to the left. But the tank will ultimately reverse its direction and move to the right and some of the water will move indefinitely to the right. And at all times the center of gravity of the system will be in the same vertical line and the system as a whole will never have any horizontal momentum.

That this is necessarily the solution is of course apparent as soon as one fixes his attention on the laws of momentum and abandons the attempt to follow intuitively and in detail the actual motion of the tank — but there would not be many paradoxes if one could only fix his attention rightly at the start. The really interesting thing about this particular paradox, however, is not its qualitative solution but the quantitive solution that follows immediately from applying the laws of mechanics to the system. Let v be the velocity of the water flowing in the tube 1234 and measured relatively to the tube. Let u be the velocity of the tank measured positively to the right. Let A be the cross-section and L the length of the tube 23. Let W be the amount of water in the tank and tubing at any instant and M the mass of the tank and tube. Then

$$(M+W)u+AL(v-u)=(M+W-AL)u+ALv$$

is the total momentum at any time of the whole containing vessel and the water contained therein but not of the water which has flowed out; the total momentum of the water which has escaped is the negative of this. The rate at which water is flowing out is Av and as the velocity of tank is u, the rate of loss of momentum from the tank and tube is Avu. Hence

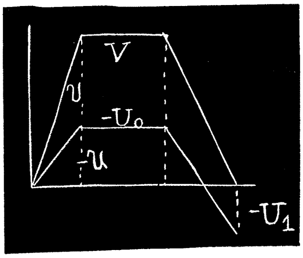

Fig. 2.

$$\frac{d}{dt}[(M+W-AL)u+ALv]=-Avu,$$

or

$$(M+W-AL)\frac{du}{dt}+AL\frac{dv}{dt}=0, \qquad (1)$$

since dW/dt, the rate of loss of water, is $-Av$. One may also write

$$(M+W)\frac{du}{dt}+AL\frac{dv'}{dt}=0, \qquad (1')$$

where $v'=v-u$ is the actual velocity of the water in the tube.

To tell just how the water would actually flow out would probably be a hydrodynamical problem incapable of solution in terms of elementary functions; but if any one of a number of reasonable assumptions be made as to the rate of change of v or v', the equations (1) or (1′) may be used to solve the motion of the tank. For instance, suppose that, when the efflux pipe is opened, the velocity v rises rapidly from 0 to a steady value V which is maintained for a while and then falls rapidly to 0 as the pipe is closed. As the rise of v is rapid, the coefficient $M+W-AL$ may be treated as constant during the rise. Hence $-u$ will rise from 0 to $-U_0$ where

$$-U_0=\frac{AL}{M+W_0-AL}, \; V \text{ and } W_0 \text{ is the initial value of } W.$$

Then the constant value U_0 will be maintained so long as V is maintained. Finally when v returns to 0 the coefficient $M+W-AL$ may again be regarded as constant and it is seen that u comes to the value

$$U_1=-\frac{AL}{M+W_0-AL}V+\frac{AL}{M+W_1-AL}V, \text{ where } W_1 \text{ is the final } W.$$

This value of u is not 0 but positive, and indeed is necessarily so in order to counterbalance the negative momentum of the water which has escaped. The figure represents the variation of v and $-u$.

Other reasonable assumptions might be made, for instance Torricelli's law of efflux. Moreover the actual displacement of the tank could be obtained by integrating u. But sufficient has been said to show how readily the problem may be treated as far as its purely mechanical side is concerned. As it is none too easy to find really good and yet essentially different problems which require for their solution merely the fundamental principles of momentum, energy, and moment of momentum, it may not be amiss to point out that the above problem may be relieved of its hydrodynamical difficulties but otherwise preserved intact by considering a small number of beads on a wire lying in a vertical plane and consisting of a series of segments at different inclinations to the horizontal.

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THE THEORY OF INVERSION AND THE QUADRATIC RECIPROCAL TRANSFORMATION.

By D. N. LEHMER, University of California.

1. The following discussion exhibits the theory of inversion in its proper light as a quadratic reciprocal transformation, and makes clear why inversion throws circles into circles but generally a curve of degree n into a curve of degree $2n$.

2. Consider a conic K and a point M. A correspondence between the points of the plane may now be set up as follows: To any point P make correspond the point P' of intersection of the line MP with the polar of P with respect to K. The point P' goes by this process into the point P again by the fundamental theorem in the theory of poles and polars: If the polar of P passes through P', the polar of P' passes through P.

If the point P moves along a straight line the point P' moves along a conic.

For P' is the locus of intersection of two projective pencils with centers at M and at the pole of the line upon which P moves. This conic thus